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Hedgehog production in spatially correlated noise*

T. Dobrowolski[a]

Institute of Physics AP, Podchorazych 2, 30-084 Cracow, Poland

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Abstract. The production of topological defects during a quench in a ϕ^4 model is investigated. The influence of a spatially correlated noise on defect production in two and three dimensions is demonstrated.

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1 Introduction

Recently, much interest has been focused on the formation of topological defects in condensed matter systems [1]. In this context, the most interesting is the creation of vortices during the transition from normal to superfluid phases of helium-4 and helium-3, and also the punching of the superconducting layers by magnetic flux tubes. The description of a continuous phase transition is particularly simple in the framework of the Landau-Ginzburg theory. An indispensable part of the description of a symmetry-breaking transition is an external noise. This work examines a dynamic transition in a ϕ^4 model in the presence of spatially correlated noise. Noises of this type are present in the description of a thermal bath of quasiparticles in superfluids [2].

The complete ϕ^4 model contains, apart from the dissipative term, also the inertia force term:

$$\partial_t^2 \phi^a(t,\mathbf{x}) + \gamma \partial_t \phi^a(t,\mathbf{x}) = \Delta \phi^a(t,\mathbf{x}) + \varepsilon \phi^a(t,\mathbf{x}) - (\phi^b \phi^b)\phi^a(t,\mathbf{x}), \quad (1)$$

where γ denotes a dissipation constant and Δ is the Laplacian operator. The indices a, b enumerate order parameters in the model. However, in most condensed matter systems the dissipative term prevails and therefore half of the modes of this system are almost unobservable in practice. These unnoticeable modes can be removed from the description by neglecting the second-order inertia term, so that condensed matter systems can be described by a much simpler overdamped (first-order) ϕ^4 model:

$$\gamma \partial_t \phi^a(t,\mathbf{x}) = \Delta \phi^a(t,\mathbf{x}) + \varepsilon \phi^a(t,\mathbf{x}) - (\phi^b \phi^b)\phi^a(t,\mathbf{x}). \quad (2)$$

The crucial characteristic of this model is provided by the shape of the potential. Depending on the sign of the parameter ε the system contains one or more ground states. This feature allows one to model the phase transition in the system.

At non-zero temperatures, the system is randomly forced by thermal fluctuations. This situation has a simple mechanical analogy, which is a Brownian motion.

The erratic motion of a Brownian particle is due to collisions with the molecules of the fluid in which it moves. These collisions allow an exchange of the energy between the fluid and the Brownian particle. If the Brownian particle is much more massive than the molecules of the fluid then the influence of the molecules on the particle can be approximated by a Gaussian noise $\eta_G(t)$:

$$m\ddot{x}(t) + \gamma \dot{x}(t) = \eta_G(t), \quad (3)$$

where $x(t)$ is the position of the Brownian particle. The generalization of Brownian motion theory to the random motion of a particle which is not necessarily heavier than the molecules of the fluid was proposed by Kubo [3]. In this case the time scale of the molecular motion is no longer much shorter than that of the motion of the particle under observation, so that the random force $\eta(t)$ can not be of Gaussian type. In addition, if we consider a stationary process we have to abandon the assumption of a constant friction and to introduce generally a frequency-dependent friction

$$m\ddot{x}(t) + \int_{t_0}^{t} \mathrm{d}t' \gamma(t-t')\dot{x}(t') = \eta(t). \quad (4)$$

In the same way non-gaussian noise can be introduced for an overdamped ϕ^4 model. In next section we consider an overdamped and retarded Landau-Ginzburg model which is defined by an integro-differential equation.

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[a] e-mail: sfdobrow@kinga.cyf-kr.edu.pl

2 Defect production

In d spatial dimensions an overdamped and retarded $O(d)$ symmetric system is described by the equation

$$\int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^d x' \gamma(t,t';\mathbf{x}-\mathbf{x}')\partial_{t'}\phi^a(t',\mathbf{x}') = \Delta\phi^a(t,\mathbf{x}) + \varepsilon(t)\phi^a(t,\mathbf{x}) - (\phi^b\phi^b)\phi^a(t,\mathbf{x}) + \eta^a(t,\mathbf{x}), \quad (5)$$

where $\eta^a(t,\mathbf{x})$ represents a noise which disturbs the system. We assume that the number of real scalar fields in the model is identical to a number of spatial dimensions *i.e.* $a = 1, 2, ..., d$. This assumption guarantees the existence of topological solutions in the model. Time dependence of the parameter ε allows for modelling a phase transition in the system. This dependence of ε on time is a consequence of change of the temperature or pressure in the system. A detailed analysis shows that ε is a function of the relative temperature $\varepsilon \sim (T_c - T)/T_c$. The dependence on time enters into this function *via* time dependence of the temperature of the system $T(t)$ or *via* time dependence of the critical temperature $T_c(t)$ which is a consequence of the pressure jump. After application of the Fourier transformation $\phi(t,\mathbf{x}) = \int \mathrm{d}^d x \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}\widetilde{\phi}(t,\mathbf{k})$, the linearized equation of motion simplify to the set of equations

$$(2\pi)^d \int_{t_0}^{t} \mathrm{d}t' \widetilde{\gamma}(t,t';\mathbf{k})\partial_{t'}\widetilde{\phi}^a(t',\mathbf{k}) = (-\mathbf{k}^2 + \varepsilon(t))\widetilde{\phi}^a(t,\mathbf{k}) + \widetilde{\eta}^a(t,\mathbf{k}). \quad (6)$$

We assume that the system is driven by a spatially correlated isotropic noise

$$\langle \widetilde{\eta}^{a*}(t,\mathbf{k})\widetilde{\eta}^b(t',\mathbf{k}')\rangle = \frac{1}{\beta}\widetilde{W}(\mathbf{k})\delta(t-t')\delta^{(d)}(\mathbf{k}-\mathbf{k}')\delta^{ab}, \quad (7)$$

where $\widetilde{W}(\mathbf{k}) = \widetilde{W}(|\mathbf{k}|)$ and $\langle ... \rangle$ denotes an average over realizations of the noise. The amplitude of the noise is connected with the temperature of the thermal bath. Although the temperature of the system changes during the transition, the most important for the formation of the kink network is the noise amplitude at the moment when the system loses its capacity to respond to changes of external parameters *i.e.* the noise amplitude at freeze-in time $\beta \equiv \beta_{\text{freeze-in}}$. The noise correlator and the friction in the system are related by the fluctuation-dissipation theorem [3]

$$\widetilde{\gamma}(t,t';\mathbf{k}) = \frac{1}{2\pi}\widetilde{W}(|\mathbf{k}|)\delta(t-t'),$$

where the temperature is absorbed in the noise correlator amplitude. In this setting, this relation ensures that the system comes to thermal equilibrium. The general solution of the linearized equations of motion is determined by a random force $\widetilde{\eta}^a$

$$\widetilde{\phi}^a(t,\mathbf{k}) = \int_{-\infty}^{t} \mathrm{d}t_1 \mathrm{e}^{\int_{t_1}^{t} \mathrm{d}t_2 \widehat{a}(t_2,\mathbf{k})}\widehat{\eta}^a(t_1,\mathbf{k}), \quad (8)$$

where $\widehat{a}(t,\mathbf{k}) \equiv \frac{-\mathbf{k}^2+\varepsilon(t)}{\widetilde{W}(\mathbf{k})}$ and $\widehat{\eta}^a(t,\mathbf{k}) \equiv \frac{\widetilde{\eta}^a(t,\mathbf{k})}{\widetilde{W}(\mathbf{k})}$ is a rescaled noise. This solution depends on the noise correlator amplitude $\widetilde{W}(\mathbf{k})$ and on the bifurcation parameter $\varepsilon(t)$. For simplicity we assume a linear time dependence of this parameter *i.e.* $\varepsilon(t) = t/\tau$, where 2τ is a quench time. It seems that any realistic time dependence of the parameter ε can be approximated by the oblique step function. We assume linear quench because for positive time we do not leave the transition area. The calculation is valid only up to a time when unstable modes start to exponentially grow, $t \le t_e < \tau$. On the other hand we know that the modulus of the parameter ε is proportional to square of the mass of the scalar field. Therefore for times proceeding transition, $t < -\tau$, a larger modulus of the parameter ε is equivalent to more massive scalar fields. As time $t \to -\infty$ the mass becomes infinite and therefore the order parameter is stabilized in its vacuum position. This behaviour agrees with initial conditions $\phi^a(t \to -\infty) = 0$.

The power spectrum for this system is defined by the equal-time correlator

$$\langle \widetilde{\phi}^{a*}(t,\mathbf{k})\widetilde{\phi}^b(t',\mathbf{k}')\rangle_{t=t'} = \mathcal{P}(t,\mathbf{k})\delta^{(d)}(\mathbf{k}-\mathbf{k}')\delta^{ab}.$$

In d dimensions the density of zeros of the scalar field is calculated with the use of the Liu-Mazenko-Halperin formula [4]

$$N = C_d \left(\frac{\int_{S_{k_m}} \mathrm{d}^d k \mathbf{k}^2 \mathcal{P}(t_e,\mathbf{k})}{\int_{S_{k_m}} \mathrm{d}^d k \mathcal{P}(t_e,\mathbf{k})} \right)^{\frac{d}{2}}. \quad (9)$$

A constant C_d depends on the number of dimensions *i.e.*

$$C_d = \begin{cases} \frac{1}{\pi}, & \text{for } d = 1 \\ \frac{1}{2\pi}, & \text{for } d = 2 \\ \frac{1}{\pi^2}, & \text{for } d = 3. \end{cases}$$

The integration is restricted to the interior of the d dimensional sphere of radius $|\mathbf{k}_m|$. A cut-off $|\mathbf{k}_m|$ separates stable and unstable modes of the system at the time $t_e = \sqrt{\widetilde{W}\tau}$, when unstable modes start to grow exponentially [5]. The critical value of momentum $|\mathbf{k}_m|$ can be identified from the linearized equations of motion whereas the explosion time t_e from the power spectrum which in the regime $t \gg \tau\mathbf{k}^2$ is given by

$$\mathcal{P}(t,\mathbf{k}) \approx \frac{1}{\beta}\sqrt{\frac{\pi\tau}{\widetilde{W}(|\mathbf{k}|)}} \exp\left(\frac{t^2}{\widetilde{W}\tau}\right) \exp\left(-\frac{2t\mathbf{k}^2}{\widetilde{W}}\right) \exp\left(\frac{\tau\mathbf{k}^4}{\widetilde{W}}\right).$$

Let us consider two opposite examples of the noise correlator amplitude $\widetilde{W}(\mathbf{k})$. In the first example an amplitude is larger for larger momentum $\widetilde{W}(\mathbf{k}) = (1 + L^2\mathbf{k}^2)^\alpha$, where α is a positive real number. In the second example an amplitude $\widetilde{W}(\mathbf{k}) = \frac{a}{\mathrm{e}^{L^n|\mathbf{k}|^n}-1}$ is smaller for larger momentum. Let us notice that in both cases the dependence of the amplitude $\widetilde{W}$ on momentum is local *i.e.* is restricted only to the interval $0 \le |\mathbf{k}| \le |\mathbf{k}_m|$ and therefore can approximate quite large class of realistic functions $\mathcal{P}(t_e,\mathbf{k})$.

I. In first example in the regime $L^2|\mathbf{k}_m|^2 \ll 1$, $\tau \leq L^4$

$$\mathcal{P}(t_e, \mathbf{k}) \approx \frac{\sqrt{\pi\tau}}{\beta} \mathrm{e}^{-2\sqrt{\tau}\mathbf{k}^2}.$$

Therefore the density of vortices ($d = 2$) is

$$N \approx \frac{1}{4\pi} \frac{1}{\tau^{\frac{1}{2}}},$$

and the density of monopoles ($d = 3$) is

$$N \approx \frac{1}{\pi^2} \sqrt{\frac{3}{4}} \frac{1}{\tau^{\frac{3}{4}}}.$$

Both predictions agree with the scaling calculated for systems driven by white Gaussian noise.

II. In second example in the regime $L|\mathbf{k}_m| \ll 1$ cut-off can be approximated by $|\mathbf{k}_m| \to \left(\frac{a}{\tau L^n}\right)^{\frac{1}{n+4}}$ and the power spectrum is the following:

$$\mathcal{P}(t_e, \mathbf{k}) \approx \frac{1}{e\beta} \sqrt{\frac{\pi\tau}{a}} |\mathbf{k}|^{\frac{n}{2}} L^{\frac{n}{2}}. \tag{10}$$

For the second class of models the density of zeros depends on the quench time and the noise characteristic length as well. In two spatial dimensions we have

$$N \approx \frac{1}{2\pi} \frac{n+4}{n+8} \left(\frac{a}{\tau L^n}\right)^{\frac{2}{n+4}}.$$

In the three-dimensional case

$$N \approx \frac{1}{\pi^2} \left(\frac{n+6}{n+10}\right)^{\frac{3}{2}} \left(\frac{a}{\tau L^n}\right)^{\frac{3}{n+4}}.$$

For $n = 0$ we recover the behaviour of the model driven by Gaussian noise, because the amplitude of the noise correlator then becomes a constant.

3 Remarks

The process of formation of topological defects, which is the main subject of this paper, is the most interesting aspect of their evolution. The number density of the topological defects is associated with the dynamics of the order parameter. As a consequence of critical slowing down the correlation length diverges, perturbations of the order parameter take longer to propagate over correlated regions and therefore it takes longer to reach an equilibrium. When the time remaining before transition equals the equilibrium relaxation time the correlation length can no longer adjust quickly enough to follow the changing temperature of the system. The same time after a quench the system regains capacity to respond to changes of external parameters. The correlation length at that time sets the characteristic length scale for the initial defect network.

The result of this paper confirms this scenario for white Gaussian noise (example II, model $n = 0$) and also for reasonable class of the Markovian noises enumerated by a positive real number (example I, $\alpha \in \mathbf{R}_+$). However there also exists noticeable class of processes (example II, $n > 0$) for which the noise imprints its own length scale on the number density of produced defects. The meaning of this result is quite intuitive. If the noise produces the zeros of the order parameter on distances smaller than the correlation length then the dynamics of the system prefer only those zeros which are separated at least by the correlation length. However, if zeros are produced by the noise on distances larger than the correlation length then the number of produced defects have to be smaller than that given by the correlation length.

Finally, it is worth stressing that so far the influence of coloured noise on defect creation and diffusion has been investigated only in case of nucleation of kinks *via* thermal activation mechanism [6] while this paper aims to study the production of topological defects during a dynamical quench.

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References

1. W.H. Zurek, Nature **317**, 505 (1993)
   Acta Phys. Polon. B **24**, 1301 (1993)
   A.J. Gill, R.J. Rivers, Phys. Rev D **51**, 6949 (1995)
   G. Karra, R.J. Rivers, Phys. Lett. B **414**, 28 (1997)
   J. Dziarmaga, P. Laguna, W. Zurek, Phys. Rev. Lett. **82**, 4749 (1999)
2. D. Vollhardt, P. Wolfle, *The superfluid phases of helium 3* (Taylor and Francis, London , 1990)
3. R. Kubo, Rep. Progr. Phys. **29**, 255 (1966)
4. B. Halperin, *Physics of Defects*, edited by R. Balian, M. Kleman, J.P. Poirier (North-Holland, New York, 1981)
   A. Weinrib, B.I. Halperin, Phys. Rev. B **26**, 1362 (1982)
   F. Liu, G.F. Mazenko, Phys. Rev. B **46**, 5963 (1992)
5. T. Dobrowolski, Europhys. Lett. **56**, 36 (2001)
6. F. Marchesoni, Phys. Lett. A **115**, 29 (1986)
   F. Marchesoni, Phys. Rev. Lett. **77**, 2364 (1996)